

# CRITERION FOR $\mathbb{Z}_d$ -SYMMETRY OF A SPECTRUM OF A COMPACT OPERATOR

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ABSTRACT. If  $A$  is a compact operator in a Banach space and some power  $A^q$  is nuclear we give a criterion of  $\mathbb{Z}_d$ -symmetry of its spectrum  $\sigma(A)$  in terms of vanishing of the traces  $\text{Trace } A^n$  for all  $n$ ,  $n \geq 0$ ,  $n \not\equiv 0 \pmod d$ , sufficiently large.

In the case of matrices, or linear operators  $T : X \rightarrow X$  in a finite-dimensional space, one can check (prove) that the following conditions are equivalent.

- (a) The spectrum of  $T$  is symmetric, or  $\mathbb{Z}_2$ -symmetric, i.e.,  $\lambda \in \sigma(T) \rightarrow -\lambda \in \sigma(T)$  and their algebraic multiplicities  $m(\lambda), m(-\lambda)$  are equal;
- (b)  $\text{Trace } T^p = 0$  for all odd  $p \in \mathbb{N}$ .

M. Zelikin [Zel08] observed and proved that this claim could be extended to  $\mathfrak{S}_1$ , the trace-class operators in a Hilbert space. We will show that such claims could be made

- (i) in general Banach spaces;
- (ii) for  $\mathbb{Z}_d$  symmetry of a spectrum,  $d \geq 2$ .

Of course, we need to make sure that  $\text{Trace}$  is well-defined if we write conditions like (b). Then, the formula for the trace  $\text{Trace } A = \sum_j \lambda_j(A)$  should be properly explained if we use it. We now recall a few notions and facts about nuclear operators (see more in [Kön86]).

An operator  $A : X \rightarrow Y$  between two Banach spaces is called *nuclear* if it has representation

$$(1) \quad Ax = \sum_{k=1}^q a_k f_k(x) y_k, \quad q \leq \infty$$

where

$$(2) \quad \begin{aligned} a_k > 0, \quad a^* = \sum a_k < \infty, \quad \text{and} \\ \|f_k|X'\| \leq 1, \quad \|y_k|Y\| \leq 1, \quad \forall k. \end{aligned}$$

A linear space of nuclear operators  $X \rightarrow Y$  is a Banach space  $\mathcal{N}(X; Y)$  with the norm

$$(3) \quad \|A\|_1 = \inf\{a^* : (1), (2)\}.$$

A linear functional *Trace* is well-defined on  $\mathcal{N}(X; X)$  (for any Banach Space  $X$ ) by

$$(4) \quad \text{Trace } A = \sum_{k=1}^q a_k f_k(y_k).$$

Of course,

$$(5) \quad \|\text{Trace } A\| \leq \|A\|_1,$$

and  $\|\text{Trace}\| = 1$ .

A. Grothendieck [Gro95] showed that for operators (1)  $[X = Y]$ ,

$$(6) \quad \text{if } \sum_{k=1}^q a_k^{2/3} < \infty$$

$$(7) \quad \text{then } \sum |\lambda_j(A)| < \infty$$

where points of the spectrum  $\sigma(A)$  are enumerated with their multiplicity, and

$$(8) \quad \text{Trace } A = \sum \lambda_j(A).$$

The presentation (4) with (1) to (3) gives a factorization

$$(9) \quad A = JF, \quad X \xrightarrow{F} \ell_2(\mathbb{N}) \xrightarrow{J} X,$$

where

$$(10) \quad Fx = \sum_1^\infty a_k^{1/2} f_k(x) e_k, \quad \text{and}$$

$$(11) \quad J\xi = \sum_1^\infty a_k^{1/2} \xi_k y_k,$$

with

$$(12) \quad \|F\| \leq (a^*)^{1/2}, \quad \|J\| \leq (a^*)^{1/2}$$

Moreover, the product  $FJ$  is a Hilbert-Schmidt operator, or of the Schatten class  $\mathfrak{S}_2$  in a Hilbert space  $\ell^2(\mathbb{N})$ ; see more in [GK69], [Sim79]. Indeed,

$$(13) \quad \langle FJ e_k, e_m \rangle = a_k^{1/2} a_m^{1/2} f_m(y_k)$$

and

$$(14) \quad \sum_{k,m=1}^{\infty} |\langle FJ e_k, e_m \rangle|^2 = \sum_{k,m=1}^{\infty} a_k a_m |f_m(y_k)|^2 \leq (a^*)^2$$

so  $\|FJ\|_2 \leq a^*$ .

By Hölder inequality for Schatten classes ([GK69] or [Sim79]),

$$(15) \quad \|BCD\|_{2/3} \leq \|B\|_2 \|C\|_2 \|D\|_2$$

so  $(FJ)^3 \in \mathfrak{S}_{2/3}$  and has a representation

$$(16) \quad (FJ)^3 = \sum_{k=1}^{\infty} c_k \langle \cdot, f_k \rangle h_k, \quad c > 0,$$

where  $\|f_k\|, \|h_k\| \leq 1$  and

$$(17) \quad \sum_{k=1}^{\infty} c_k^{2/3} < \infty.$$

Therefore,

$$(18) \quad A^4 = J(FJ)^3 F = \sum_{k=1}^{\infty} c_k \langle F(\cdot), f_k \rangle J h_k$$

has  $\frac{2}{3}$ -property (6) and

$$(19) \quad \sum_{j=1}^{\infty} |\lambda_j(A^q)| < \infty \text{ for all } q \geq 4,$$

with

$$(20) \quad \text{Trace } A^q = \sum_{j=1}^{\infty} \lambda_j(A^q)$$

More careful geometric analysis, based on approximative characteristics of operators [MP66], [Pie87] — if we use [Kön80], or [Kön86, Theorem 4.a.6, p. 227] — shows that we can lower  $q$  in (19), (20) to 3. Indeed,  $(FJ)^2$  is in  $\mathfrak{S}_1(\ell^2(\mathbb{N}))$ , so there are finite-dimensional operators  $G_n$ ,  $\text{Rank } G_n \leq n$ , such that

$$(21) \quad \sum_n \alpha_n < \infty, \quad \text{where } \alpha_n := \|(FJ)^2 - G_n\|$$

Then

$$(22) \quad \|A^3 - JG_n F\| \leq a^* \cdot \alpha_n$$

and by [Kön86, Theorem 4.a.6]

$$(23) \quad A^3 \text{ is nuclear,}$$

$$(24) \quad \sum_j |\lambda_j(A^3)| \leq 2a^* \sum_n \alpha_n < \infty,$$

and

$$(25) \quad \text{Trace } A^3 = \sum_j \lambda_j(A^3).$$

But this remark will not improve our Theorem 1 (below) in an essential way (just in (41) we can say  $p \geq p_* \geq 3q^*$ ).

In a Hilbert space  $X = H$  by Lisdkiĭ Theorem [Lid59], for any trace-class operator  $C \in \mathfrak{S}_1$ ,

$$(26) \quad \sum_{j=1}^{\infty} |\lambda_j(C)| < \infty$$

and

$$(27) \quad \text{Trace } C = \sum_{j=1}^{\infty} \lambda_j(C).$$

Maybe, talking just about nuclear operators, M. Zelikin considered in [Zel08, Thm. 2] only Hilbert spaces.

Before stating our main result let us recall [DS58, Chapter VII, Sections 3 and 4] elements of Riesz theory of compact operators.

If  $T : X \rightarrow X$  is compact its spectrum  $\sigma(T)$  is discrete with 0 being the only accumulation point, and it has the following properties

- (i) for any  $\rho > 0$ ,  $\sigma(T) \cap \{z : |z| \geq \rho\}$  is a finite set;
- (ii) if

$$(28) \quad \delta(\alpha) = \frac{1}{2} \min\{|\alpha - \lambda| : \lambda \in \sigma(T), \lambda \neq \alpha\}$$

[so  $\delta(\alpha) > 0$  for any  $\alpha \in \mathbb{C} \setminus 0$ ] and

$$(29) \quad P(\alpha) = \frac{1}{2\pi i} \int_{|z-\alpha|=\delta(\alpha)} (z - T)^{-1} dz,$$

then

$$(30) \quad m(\alpha) = \text{Rank } P(\alpha) < \infty, \quad \alpha \in \mathbb{C} \setminus \{0\}$$

with

$$(31) \quad m(\alpha) = 0 \quad \text{if and only if } \alpha \notin \sigma(T).$$

For  $\alpha \in \sigma(T) \setminus 0$ ,  $m(\alpha)$  is an algebraic multiplicity of an eigenvalue  $\alpha$ .

The operational calculus [DS58, Chapter VII, Sections 3 and 4] explains that for any  $\rho > 0$  such that

$$(32) \quad \sigma(T) \cap \{|z| = \rho\} = \emptyset$$

we have

$$(33) \quad T = \sum_{|\alpha| > \rho} T(\alpha) + S, \quad \text{where } T(\alpha) = \frac{1}{2\pi i} \int_{|z-\alpha|=\delta(\alpha)} z(z-T)^{-1} dz$$

is an operator of rank  $m(\alpha)$  with

$$(34) \quad \sigma(T(\alpha)) = \{\alpha\},$$

and

$$(35) \quad S = \frac{1}{2\pi i} \int_{|z|=\rho} z(z-T)^{-1} dz.$$

Moreover, for any entire function  $F(z)$ , say, for polynomials,

$$(36) \quad F(T) = \sum_{|\alpha| > \rho} F(T(\alpha)) + F(S),$$

where by the Riesz-Cauchy formulae,

$$(37) \quad F(T(\alpha)) = \frac{1}{2\pi i} \int_{|z-\alpha|=\delta(\alpha)} F(z)(z-T)^{-1} dz, \quad F(S) = \frac{1}{2\pi i} \int_{|z|=\rho} F(z)(z-T)^{-1} dz.$$

It follows that

$$(38) \quad \text{Trace } F(T(\alpha)) = F(\alpha) \cdot m(\alpha).$$

$$(39) \quad F(T(\alpha)) = 0 \quad \text{if } F^{(j)}(\alpha) = 0, \quad 0 \leq j \leq m(\alpha).$$

Now we are ready to prove

**Theorem 1.** *Let  $T$  be a compact operator in a Banach space  $X$ , and some power  $T^{q*}$  is a nuclear operator. Then  $\sigma(T)$  is  $\mathbb{Z}_d$ -symmetric, i.e., for any  $\beta \in \mathbb{C} \setminus \{0\}$ ,*

$$(40) \quad m(\beta\omega^k) = m(\beta) \text{ for all } k = 0, 1, \dots, d-1, \quad \omega = \exp\left(i\frac{2\pi}{d}\right)$$

*if and only if*

$$(41) \quad \text{Trace } T^{dp+r} = 0, \quad 1 \leq r \leq d-1,$$

*for all sufficiently large  $p$ , say  $p \geq p_* \geq 4q_*$ .*

Of course, if  $d = 2$ , this is an extension of [Zel08], Thm. 2, to a Banach case.

*Proof. Part 1:* (40)  $\Rightarrow$  (41). This is an “algebraic” claim although first we notice: the assumption  $p \geq 4q_*$  guarantees that all operators  $T^n$ ,  $n = dp+r$ , in (41) satisfy  $\frac{2}{3}$ -condition so by Grothendieck theorem

$$(42) \quad \text{Trace } T^n = \sum_{j=1}^{\infty} \lambda_j(T^n)$$

and the absolute convergence permits to rearrange the terms of the right sum as we wish to write

$$(43) \quad \text{Trace } T^n = \sum \mu \cdot m(\mu; T^n)$$

With

$$(44) \quad m(\mu; T^n) = 0 \text{ for } \mu \notin \sigma(T^n)$$

we can “add” the terms with  $\mu \notin \sigma(T^n)$  and this does not change the right side in (43). For

$$(45) \quad n = dp + r \in (41) \text{ define } g = \gcd\{r, d\}$$

so

$$(46) \quad r = ag, \quad d = bg, \quad (a, b) = 1$$

and with  $r \leq d-1$  we have  $1 \leq a < b$ . For any  $\mu \in \mathbb{C} \setminus \{0\}$  take its  $\mathbb{Z}_b$ -orbit, i.e.,

$$(47) \quad \tilde{\mu} = \{\mu \cdot \tau^j : 0 \leq j < b\}, \quad \tau = \omega^q = \exp\left(i\frac{2\pi}{b}\right).$$

The sum in (43) could be written as

$$(48) \quad \sum_{\mathbb{Z}_b\text{-orbits}} \sum_{j=0}^{b-1} \mu \tau^j \cdot m(\mu \tau^j; T^n)$$

where for certainty  $\mu$  in the orbit (47) is chosen as  $\mu = |\mu|e^{i\vartheta}$ ,  $0 \leq \vartheta < \frac{2\pi}{b}$ . Now we will show that the sum in (48) over each orbit is equal to zero. With numbers as in (45) put  $\kappa = \exp\left(i\frac{2\pi}{n}\right)$  so  $\kappa^n = 1$  and notice that if  $\mu = \lambda^n$ , we choose

$$(49) \quad \lambda = |\mu|^{1/n} e^{i\vartheta'}, \quad \vartheta' = \frac{\vartheta}{n},$$

then

$$(50) \quad \left(\lambda \omega^k\right)^n = \mu \omega^{k(dp+r)} = \mu \omega^{kr} = \mu \tau^{ak}$$

and

$$(51) \quad \begin{aligned} \sum_{j=0}^{b-1} \mu \tau^j m(\mu \tau^j; T^n) &= \frac{1}{g} \sum_{k=0}^{d-1} \mu \tau^{ak} m(\mu \tau^{ak}; T^n) \stackrel{(1)}{=} \\ &= \frac{1}{g} \sum_{k=0}^{d-1} \left(\lambda \omega^k\right)^n \sum_{s=0}^{n-1} m(\lambda \omega^k \kappa^s; T) \stackrel{(2)}{=} \\ &= \frac{1}{g} \sum_{s=0}^{n-1} \sum_{k=0}^{d-1} \left(\lambda \omega^k\right)^n m(\lambda \kappa^s \cdot \omega^k; T) \stackrel{(3)}{=} \\ &= \frac{1}{g} \sum_{s=0}^{n-1} m(\lambda \kappa^s; T) \mu \sum_{k=0}^{d-1} \tau^{\alpha k} \stackrel{(4)}{=} \\ &\quad \mu \left( \sum_{s=0}^{n-1} m(\lambda \kappa^s; T) \right) \left( \sum_{j=0}^{b-1} \tau^j \right) \stackrel{(5)}{=} 0 \end{aligned}$$

The steps in (51) are justified in the following way. (1) comes from (50). (2) is just the change of order of the double summation. (3) uses in essential way the theorem's assumption (40) on  $m(\beta \omega^k)$  being independent on  $k$ . (4) is bases on the properties of the roots  $\omega$ ,  $\tau$ ,  $\omega^d = 1$ ,  $\tau = \omega^g$  under (46). Of course, in (5)  $\sum_{j=0}^{b-1} \tau^j = 0$ , and  $\{\tau^{ak}\}_{k=0}^{d-1}$  runs  $g$  times over  $\{\tau^j\}_{j=0}^{b-1}$ . Part (40)  $\Rightarrow$  (41) is proven.  $\square$

*Proof. Part 2:* (41)  $\Rightarrow$  (40). Take  $\lambda \neq 0$  and as before

$$(52) \quad n = dp_* + dp + r, \quad 1 \leq r \leq d-1, \quad p \geq 0$$

and  $0 < \rho < |\lambda|$  is such that

$$(53) \quad \sigma(T) \cap \{z \in \mathbb{C} : |z| = \rho\} = \emptyset,$$

with

$$(54) \quad \tilde{\lambda} = \{\lambda\omega^k : 0 \leq k \leq d-1\}$$

being the  $\mathbb{Z}_d$ -orbit of  $\lambda$ . Now we use (36) for the special choice  $F = F_{pr}$  with

$$(55) \quad F_{pr}(z) = \left(\frac{z}{\lambda}\right)^{dp_*+dp+r} \varphi(z),$$

where

$$(56) \quad \varphi(z) = \prod_{\substack{|\alpha| \geq \rho \\ \alpha \in \sigma(T) \\ \alpha \notin \tilde{\lambda}}} \left( \frac{z^d - \alpha^d}{\lambda^d - \alpha^d} \right)^{m(\alpha)} =$$

$$(57) \quad = \psi(z^d), \quad \text{and } \psi \text{ is a polynomial.}$$

Then by (39)

$$(58) \quad \varphi(T(\alpha)) = 0,$$

$$(59) \quad F_{pr}(T(\alpha)) = 0, \quad \forall \alpha \notin \tilde{\lambda}, \quad |\alpha| > \rho$$

but for  $\beta \in \tilde{\lambda}$ , i.e.,  $\beta = \lambda\omega^k$ ,

$$(60) \quad \text{Trace } F_{pr}(T(\beta)) = m(\beta)F_{pr}(\beta) = m(\lambda\omega^k)\omega^{kr}.$$

Therefore,

$$(61) \quad \text{Trace } F_{pr}(T) = \sum_{k=0}^{d-1} \omega^{kr} m(\lambda\omega^k) + \text{Trace } F_{pr}(S)$$

where

$$(62) \quad F_{pr}(S) = \left(\frac{T}{\lambda}\right)^{dp_*} \cdot \frac{1}{2\pi i} \int_{|z|=\rho} \left(\frac{z}{\lambda}\right)^{dp+r} \varphi(z)(z-T)^{-1} dz.$$

Put

$$(63) \quad \Phi = \max\{|\varphi(z)| : |z| \leq \rho\}$$

and with (53)

$$(64) \quad M = \max\{\|R(z; T)\| : |z| = \rho\} < \infty$$

Then

$$(65) \quad \|F_{pr}(S)\|_1 \leq Ct^p, \quad \text{any } r, \quad 1 \leq r \leq d-1,$$

where

$$(66) \quad C = \frac{\Phi \cdot M \cdot \rho \cdot \|T^{dp*}\|_1}{|\lambda|^{dp*}}$$

and

$$(67) \quad t = \left(\frac{\rho}{|\lambda|}\right)^d < 1.$$

Now by (41) and (61)

$$(68) \quad 0 = \sum_{k=0}^{d-1} \omega^{kr} m(\lambda \omega^k) + \xi_{pr} \quad \text{for any } p \geq 1 \text{ and } r, \quad 1 \leq r \leq d-1.$$

The sum  $\sum_{k=0}^{d-1}$  does not depend on  $p$  but the remainder by (65) to (67) have estimates

$$(69) \quad |\xi_{pr}| \leq Ct^p \quad \text{so} \quad \xi_{pr} \rightarrow 0 \quad (p \rightarrow \infty)$$

This implies by (68)

$$(70) \quad \sum_{k=0}^{d-1} \omega^{kr} m(\lambda \omega^k) = 0, \quad \forall r, \quad 1 \leq r \leq d-1$$

or

$$(71) \quad y_k = m(\lambda \omega^k), \quad 1 \leq k \leq d-1.$$

is a solution of the system

$$(72) \quad \sum_{k=1}^{d-1} \omega^{kr} y_k = -y_0, \quad 1 \leq r \leq d-1.$$

Its determinant is of Vandermonde type so

$$(73) \quad \det\{\omega^{kr}\}_{k,r=1}^{d-1} \neq 0,$$

and the identities

$$(74) \quad \sum_{k=0}^{d-1} (\omega^r)^k = 0, \quad \forall r, \quad 1 \leq r \leq d-1$$

show that by (72)

$$(75) \quad y_k = y_0, \text{ i.e., } m(\lambda \omega^k) = m(\omega), \forall k, \quad 1 \leq k \leq d-1.$$

This proves that the multiplicity function  $m$  is constant on  $\mathbb{Z}_d$ -orbits in  $\mathbb{C} \setminus \{0\}$ , and (40) is proven.  $\square$

It is worth to notice that the proof of Part II does not use any form of Grothendick or Lidskii theorem but it uses only properties of a linear function  $Trace$  on  $\mathcal{N}(X; X)$  and an elementary formula for  $Trace K$  when  $K$  is an operator of finite rank.

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